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Exactly solvable three-body systems with internal degrees of freedom

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Abstract

The approach of multi-dimensional SUSY quantum mechanics is used in an explicit construction of exactly solvable three-body (and quasi-exactly-solvable N-body) matrix problems on a line. From intertwining relations with time-dependent operators, we build exactly solvable non-stationary scalar and 2×2 matrix three-body models which are time-dependent extensions of the Calogero model. Finally, we investigate the invariant operators associated with these systems.

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1. Introduction

During the last three decades exactly solvable N-body problems have provided useful tools to investigate formal algebraic properties with applications to different branches of physics. The most widely studied model is the so-called Calogero model [1] and its various generalizations which essentially are many-body extensions of the one-dimensional singular harmonic oscillator model. Calogero-like models have been developed incorporating different root systems [2], q-deformations [3], PT-symmetric generalizations [4], many-body forces [5], multi-dimensions [6] and internal degrees of freedom with potentials which couple them (matrix potentials) [7]. Even if coupled channel problems in general have a continuum [8] and discrete spectrum, in the Calogero-like models (with harmonic attraction) the spectrum is purely discrete, reflecting an essentially confining dynamics. Physical applications of this dynamics have been elaborated in the context of localized systems such as, in particular, Paul traps and quantum dots [9].

An important generalization of the Calogero model is its supersymmetrization [10]. Supersymmetric quantum mechanics (SUSY QM) [11] is a suitable framework to discover and investigate one-dimensional, one-particle exactly solvable models [12]. The same strategy applies to systems with multiple degrees of freedom. Multi-dimensional SUSY QM, first constructed in [13], leads to a superHamiltonian which includes a chain of matrix Hamiltonians (cf coupled channels or internal degrees of freedom, like spin [14]). Supersymmetry ensures that the spectral properties and the eigenfunctions of the Hamiltonians belonging to the chain are algebraically interrelated. This analysis can be reinterpreted with reference to a one-dimensional multi-particle problem enlarging classes of many-body exactly solvable problems in a similar way as for the one-dimensional one-particle problems [15].

We start from the superpotential of the Calogero system which corresponds to two exactly solvable scalar components of the superHamiltonian. They are intertwined to the neighbouring matrix components of the superHamiltonian. This implies that a part of the spectrum for both matrix potentials and the corresponding wavefunctions are known. Thus from solvable scalar models by supersymmetric techniques quasi-exactly-solvable matrix problems are generated. This approach generates matrix N-particle models which can be considered in the context of recently constructed scalar quasi-exactly-solvable [16] and so-called partially solvable [17] models.

In section 2 we review the basic aspects of multi-dimensional SUSY QM [13] and introduce its reinterpretation in terms of multi-particle one-dimensional SUSY QM (see details in this paper [15]). In particular, we focus attention on models with exactly solvable scalar components of the superHamiltonian.

In section 3 starting from the Calogero model [1] we analyze its SUSY extension which includes quasi-exactly-solvable [18] N-particle matrix models. Furthermore, we study three-body problems in detail because the properties of the chain of the components of the superHamiltonian simplify considerably so that the spectrum and the wavefunctions for the (only) matrix Hamiltonian are fully determined from those of (two) scalar Hamiltonians.

In addition to exactly solvable stationary problems we also consider time-dependent potentials and, correspondingly, exactly solvable time-dependent problems. In the context of one-dimensional one-particle SUSY QM (and Darboux transformations) such problems were investigated in [19]. In non-stationary SUSY QM supercharges (of first and second order in space derivatives) commute with the non-stationary Schrödinger superoperator and intertwine consecutive components of the supersymmetric chain. Following methods developed in recent investigations of the time-dependent harmonic oscillator model and its generalizations [20], in section 4 we construct time-dependent three-particle solvable problems. In this section we achieve our main goal after having prepared the relevant framework in the previous sections. These results can be interpreted as a time-dependent generalization of the SUSY Calogero model, which can be shown to be solvable by introducing unitary intertwining operators (non-polynomial in derivatives). An extension of this method to the N-body Calogero model described in section 2 leads to time-dependent quasi-exactly-solvable matrix models. While the Calogero model is a many-body generalization of the 'singular' harmonic oscillator model, its time-dependent version, which we study, are correspondingly extensions of the oscillator problem with time-dependent parameters. This last problem has attracted much interest in the literature [20, 21] and has applications in different areas of physics.

2. Multi-dimensional SUSY QM and N-particle quasi-exactly-solvable stationary problems

The supersymmetric quantum system for an arbitrary number of dimensions N consists [13] of the superHamiltonian H_S and the supercharges $Q^{\pm}$ with the algebra (SUSY QM algebra):

$$H_S = \{Q^+, Q^-\} \tag{1}$$

$$(Q^+)^2 = (Q^-)^2 = 0 \tag{2}$$

$$[H_S, Q^{\pm}] = 0. \tag{3}$$

An explicit realization is given by[3]

$$H_S = \tfrac{1}{2}\left(-\Delta + \sum_{i=1}^{N} (\partial_i W)^2 - \Delta W \right) + \sum_{i,j=1}^{N} \psi_i^+ \psi_j \partial_i \partial_j W$$
$$\Delta \equiv \sum_{i=1}^{N} \partial_i \partial_i \qquad \partial_i \equiv \partial/\partial x_i \tag{4}$$

$$Q^\pm \equiv \frac{1}{\sqrt{2}} \sum_{j=1}^{N} \psi_j^\pm (\pm\partial_j + \partial_j W) \tag{5}$$

where ψ_i, ψ_i^+ are standard fermionic operators:

$$\{\psi_i, \psi_j\} = 0 \qquad \{\psi_i^+, \psi_j^+\} = 0 \qquad \{\psi_i, \psi_j^+\} = \delta_{ij}. \tag{6}$$

The dynamics of a particular SUSY QM system is determined by a superpotential W, which depends on N coordinates $(x_1, \ldots, x_N)$.

In general, solvable scalar models in multi-dimensional quantum mechanics for one particle admit simple separation of variables and are, therefore, reducible to one-dimensional problems. For this reason, from now on we will alternatively interpret multi-dimensional SUSY QM as a multi-particle problem on a line because one knows classes of solvable models (the Calogero model, Sutherland model and others [2]) which do not admit such a straightforward separation.

For N-particle systems on a line it is natural to consider [15] superpotentials with a separable centre-of-mass motion (CMM), satisfying the condition:

$$W(x_1, \ldots, x_N) = w(x_1, \ldots, x_N) + W_0(x_1 + \cdots + x_N)$$
$$\sum_{j=1}^{N} \partial_j w(x_1, \ldots, x_N) = 0 \tag{7}$$

i.e. the first term $w(x_1, \ldots, x_N)$ does not depend on $\sum_{i=1}^{N} x_i$. We will restrict ourselves to superpotentials (7) with $W_0 = 0$ or, equivalently, $\sum_{k=1}^{N} \partial_k W(x_1, \ldots, x_N) = 0$.

For the superpotentials (7) one can use the well known Jacobi coordinates[4] [22]

$$y_b = \frac{1}{\sqrt{b(b+1)}}(x_1 + \cdots + x_b - b x_{b+1})$$
$$y_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i \tag{8}$$

or, briefly, $y_k = \sum_{l=1}^{N} R_{kl} x_l$, where the orthogonal matrix R is determined by (8). For the supersymmetric systems we also introduce the fermionic analogues[5] of the Jacobi variables:

$$\phi_b = \frac{1}{\sqrt{b(b+1)}}(\psi_1 + \cdots + \psi_b - b\psi_{b+1})$$
$$\phi_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \psi_i$$

which satisfy the canonical anticommutation relations:

$$\{\phi_k, \phi_l\} = 0 \qquad \{\phi_k^+, \phi_l^+\} = 0 \qquad \{\phi_k, \phi_l^+\} = \delta_{kl}. \tag{9}$$

[3] Here and below the indices $i, j, k, \ldots$ range from 1 to N.

[4] From this moment on, the variables denoted by letters $a, b, c, \ldots$ range from 1 to $(N-1)$.

[5] The use of these variables has been instrumental [15] in clarifying the role of the permutation group S_N in SUSY QM.

In terms of the Jacobi variables the supercharges (5) can be rewritten as

$$Q^{\pm} = q^{\pm} \pm \frac{1}{\sqrt{2}}\phi_N^{\pm}\frac{\partial}{\partial y_N}$$
$$q^{\pm} \equiv \frac{1}{\sqrt{2}}\sum_{b=1}^{N-1}\phi_b^{\pm}\left(\pm\frac{\partial}{\partial y_b} + \frac{\partial}{\partial y_b}w\right).$$

Because

$$\left\{q^{\pm}, \phi_N^{\mp}\frac{\partial}{\partial y_N}\right\} = 0 \tag{10}$$

the free motion of the centre-of-mass in the superHamiltonian can be separated:

$$H_{\mathrm{S}} = \{Q^+, Q^-\} \equiv h - \frac{1}{2}\frac{\partial^2}{\partial y_N^2} \tag{11}$$

where

$$h \equiv \{q^+, q^-\} = \frac{1}{2}\sum_{b=1}^{N-1}\left(-\frac{\partial^2}{\partial y_b^2} + \left(\frac{\partial w}{\partial y_b}\right)^2 - \frac{\partial^2 w}{\partial y_b^2}\right) + \sum_{b,c=1}^{N-1}\phi_b^+\phi_c\frac{\partial^2 w}{\partial y_b \partial y_c} \tag{12}$$

is a $(N-1)$-dimensional superHamiltonian expressed in Jacobi variables $y_1, \ldots, y_{N-1}$. In the following we will consider only this reduced superHamiltonian h.

The operator h acting in the fermionic Fock space:

$$\phi_{b_1}^+ \ldots \phi_{b_M}^+|0\rangle \qquad M < N \qquad b_i < b_j \qquad \text{for} \quad i < j \tag{13}$$

generated by fermionic creation operators ϕ_b^+, conserves the corresponding fermionic number:

$$[h, N_{\mathrm{F}}] = 0 \qquad \text{with} \quad N_{\mathrm{F}} \equiv \sum_{b=1}^{N-1}\phi_b^+\phi_b. \tag{14}$$

Therefore, in the basis (13) it has [13] a block-diagonal form:

$$h = \begin{pmatrix} h^{(0)} & 0 & \cdots & 0 & 0 \\ 0 & h^{(1)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & h^{(N-2)} & 0 \\ 0 & 0 & 0\cdots & 0 & h^{(N-1)} \end{pmatrix} \tag{15}$$

where the matrix operators $h^{(M)}$ of dimension $C_{N-1}^M \times C_{N-1}^M$ are the projections of h onto the subspaces with fixed fermionic number $N_{\mathrm{F}} = M$. These components are standard Schrödinger operators with matrix potentials and can be obtained from (12) by a suitable matrix realization of the fermionic variables ϕ_b.

The supercharge q^+ increases the fermionic number from M to $M+1$ and has the under-diagonal structure:

$$q^+ = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ q_{(0,1)}^+ & 0 & \cdots & 0 & 0 \\ 0 & q_{(1,2)}^+ & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & q_{(N-2,N-1)}^+ & 0 \end{pmatrix}. \tag{16}$$

Similarly, $q^- = (q^+)^{\dagger}$ is an over-diagonal matrix operator with nonzero elements $q_{(M+1,M)}^- = (q_{(M,M+1)}^+)^{\dagger}$.

Superinvariance (3) of the superHamiltonian corresponds, in components, to the intertwining relations:

$$hq^+ = q^+h \Leftrightarrow h^{(M+1)}q^+_{(M,M+1)} = q^+_{(M,M+1)}h^{(M)}$$
$$q^-h = hq^- \Leftrightarrow q^-_{(M+1,M)}h^{(M+1)} = h^{(M)}q^-_{(M+1,M)}.$$

These relations lead [13] to important connections between spectra and eigenfunctions of 'neighbouring' Hamiltonians, with fermionic numbers differing by 1. In particular, $q^+_{(M,M+1)}$ maps eigenfunctions of $h^{(M)}$ onto those of $h^{(M+1)}$ with the same energy E_K:

$$\Psi^{M+1}_K(\vec{y}) = q^+_{(M,M+1)}\Psi^M_K(\vec{y}) \qquad h^{(M)}\Psi^M_K(\vec{y}) = E_K\Psi^M_K(\vec{y}). \tag{17}$$

Analogously, $q^-_{(M,M-1)}$ maps eigenfunctions of $h^{(M-1)}$ onto those of $h^{(M)}$ with the same value of energy (see details in [13]).

In particular, the spectrum of the matrix $(N-1)\times(N-1)$ Hamiltonian $h^{(1)}_{ik}$ consists of two portions, one of which coincides with the spectrum of the scalar Hamiltonian $h^{(0)}$. Thus if the scalar problem with $h^{(0)}$ is solvable the matrix problem with $h^{(1)}_{ik}$ becomes quasi-exactly-solvable [18]. Similarly, the matrix Hamiltonian $h^{(N-2)}$ is also quasi-exactly-solvable provided the last (scalar) Hamiltonian $h^{(N-1)}$ is exactly solvable.

3. Stationary solutions of three-body problem with internal degrees of freedom

As a realization of what we presented in section 2, we provide an explicit construction for the N-body Calogero model. Substituting the superpotential[6] which depends only on the first $(N-1)$ bosonic Jacobi coordinates $y_1, y_2, \ldots, y_{N-1}$:

$$W(x_1, x_2, \ldots, x_N) = \alpha\sum_{i\neq j=1}^{N}(x_i - x_j)^2 + \frac{\gamma}{2}\sum_{i\neq j=1}^{N}\ln|x_i - x_j| = w(y_1, y_2, \ldots, y_{N-1}) \tag{18}$$

into (4), after some manipulations we obtain apart from a constant energy shift:

$$H_{\mathrm{S}} = -\frac{1}{2}\Delta^{(N)} + 4\alpha^2N\sum_{i\neq j=1}^{N}(x_i - x_j)^2 + \frac{1}{2}\gamma(\gamma+1)\sum_{i\neq j=1}^{N}\frac{1}{(x_i - x_j)^2}$$
$$+\gamma\sum_{i\neq j=1}^{N}\psi_i^\dagger\psi_j\frac{1}{(x_i - x_j)^2} - \gamma\sum_{i\neq j=1}^{N}\psi_i^\dagger\psi_i\frac{1}{(x_i - x_j)^2}. \tag{19}$$

After subtraction of the free CMM (11) one obtains[7] a reduced Hamiltonian h from (12), with the superpotential $w(y_1, y_2, \ldots, y_{N-1})$. The expression for scalar $h^{(0)}$ can be derived from the superHamiltonian (19) by taking into account that the fermionic terms vanish in the subspace with $N_{\mathrm{F}} = 0$:

$$h^{(0)} = -\frac{1}{2}\Delta^{(N-1)}_y + 4\alpha^2N\sum_{i\neq j=1}^{N}(x_i - x_j)^2 + \frac{1}{2}\gamma(\gamma+1)\sum_{i\neq j=1}^{N}\frac{1}{(x_i - x_j)^2}. \tag{20}$$

It corresponds to the well known exactly solvable N-body Calogero model [1, 5]. As was discussed at the end of the previous section, the matrix Hamiltonian $h^{(1)}_{ik}$ is thus quasi-exactly-solvable and the associated part of its energy levels coincides with the oscillator-like spectrum of (20).

[6] This form for W is suggested by the ground state wavefunction of the conventional Calogero model (see (20)) and it is known to be a particular choice among possible alternatives.

[7] From now on we will use the notation ∂_i for $\partial/\partial y_i$.

The last scalar component $h^{(N-1)}$ of the superHamiltonian (19) is obtained by its reduction to the subspace of (13) with maximal fermionic occupation number $N_{\mathrm{F}} = (N-1)$. Only the last fermionic term in (19) is effective and $h^{(N-1)}$ coincides[8] with $h^{(0)}$ after the γ into $(-\gamma)$ replacement[9]. It is clear that exact solvability of $h^{(N-1)}$ leads again to quasi-exact-solvability of the matrix Hamiltonian $h^{(N-2)}$.

For $N=4$ the chain of (15) consists of two scalar Calogero Hamiltonians $h^{(0)}$, $h^{(3)}$ and two matrix 3×3 Hamiltonians $h^{(1)}_{ik}$ and $h^{(2)}_{ik}$, where for example [13]

$$h^{(1)}_{ik} = -\tfrac{1}{2}\Delta^{(3)}_y + \tfrac{1}{2}(\partial_i w)^2 + \tfrac{1}{2}\begin{pmatrix} (\partial_1^2-\partial_2^2-\partial_3^2)w & 2\partial_1\partial_2 w & 2\partial_1\partial_3 w \\ 2\partial_1\partial_2 w & (\partial_2^2-\partial_1^2-\partial_3^2)w & 2\partial_2\partial_3 w \\ 2\partial_1\partial_3 w & 2\partial_2\partial_3 w & (\partial_3^2-\partial_1^2-\partial_2^2)w \end{pmatrix} \tag{21}$$

and $h^{(2)}_{ik}$ has a similar structure. The Hamiltonian (21) is intertwined to $h^{(0)}$ by $q^+_{(0,1)} \equiv (A_1^-, A_2^-, A_3^-)$, where $A_i^- = (A_i^+)^\dagger \equiv \frac{1}{\sqrt{2}}(\partial_i + \partial_i w(y_1, y_2, y_3))$. Therefore since $h^{(0)}$ is solvable, $h^{(1)}_{ik}$ is quasi-exactly-solvable. Similar considerations hold concerning the intertwining of $h^{(2)}_{ik}$ and $h^{(3)}$. The 'non-quasi-exactly-solvable' portions of $h^{(1)}_{ik}$ and $h^{(2)}_{ik}$ coincide [13] because of an additional intertwining between them.

It is clear that, when the matrix operator $h^{(1)}$ happens to coalesce with $h^{(N-2)}$, the quasi-exactly-solvable matrix problem becomes exactly solvable. This is the case for the $N=3$ Calogero model. We now consider the standard Calogero Hamiltonian for three particles on a line with repulsive singular terms. In terms of Jacobi coordinates

$$y_1 = \frac{x_1 - x_2}{\sqrt{2}} \qquad y_2 = \frac{x_1 + x_2 - 2x_3}{\sqrt{6}}$$

the superpotential $w(y_1, y_2)$ up to an irrelevant constant has the form

$$w(y_1, y_2) = 6\alpha(y_1^2 + y_2^2) + \gamma \ln \left| y_1 \left(\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2\right)\left(-\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2\right)\right|. \tag{22}$$

The Hamiltonian (20) can be rewritten as

$$h^{(0)} = -\frac{1}{2}(\partial_1^2 + \partial_2^2) + \frac{1}{2}\gamma(\gamma+1)\left\{\frac{1}{y_1^2} + \frac{1}{(\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2)^2} + \frac{1}{(-\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2)^2}\right\}$$
$$+72\alpha^2(y_1^2 + y_2^2) \tag{23}$$

or, equivalently:

$$h^{(0)} = A_1^+ A_1^- + A_2^+ A_2^- \tag{24}$$

where (A_1^-, A_2^-) are the components of the vector operator

$$q^+_{(0,1)} \equiv (A_1^-, A_2^-) \tag{25}$$

which can be expressed in terms of the superpotential as

$$A_1^- = (A_1^+)^\dagger \equiv \frac{1}{\sqrt{2}}(\partial_1 + \partial_1 w(y_1, y_2))$$
$$\equiv \frac{1}{\sqrt{2}}\left(\partial_1 + 12\alpha y_1 + \frac{\gamma}{2}\left[\frac{2}{y_1} + \frac{1}{\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2} - \frac{1}{-\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2}\right]\right)$$

[8] Let us note that the eigenfunctions of $h^{(0)}$ and $h^{(N-1)}$ are not connected directly by supercharges $q^\pm$, contrary to the hypothesis of [23] in the context of Calogero-like models. In this connection, it was recently found [24] that their eigenfunctions are related by a Dunkl-like differential operator.

[9] Note that the ground state energy of the Calogero model depends on γ.

$$A_2^- = (A_2^+)^\dagger \equiv \frac{1}{\sqrt{2}}(\partial_2 + \partial_2 w(y_1, y_2))$$
$$\equiv \frac{1}{\sqrt{2}}\left(\partial_2 + 12\alpha y_2 + \frac{\sqrt{3}\gamma}{2}\left[\frac{1}{\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2} + \frac{1}{-\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2}\right]\right).$$

The Hamiltonian $h^{(0)}$ is not symmetric under the exchange of variables y_1, y_2. However, its wavefunctions can be obtained from the well known wavefunctions of the Calogero Hamiltonian [1], which are symmetric under the permutations of x_i $(i = 1, 2, 3)$.

According to section 2, the Hamiltonian $h^{(0)}$ generates a chain which includes a second scalar Hamiltonian defined (apart from a constant) by

$$h^{(2)} = B_1^+B_1^- + B_2^+B_2^- = -\tfrac{1}{2}(\partial_1^2 + \partial_2^2) + \tfrac{1}{2}\gamma(\gamma - 1)$$
$$\times\left\{\frac{1}{y_1^2} + \frac{1}{(\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2)^2} + \frac{1}{(-\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2)^2}\right\} + 72\alpha^2(y_1^2 + y_2^2) \tag{26}$$

where we have introduced the operators $B_l^\pm \equiv \epsilon_{lk}A_k^\mp$, $\epsilon_{12} = -\epsilon_{21} = 1$ and $\epsilon_{11} = \epsilon_{22} = 0$.

Also included in the chain is the 2×2 matrix Hamiltonian:

$$h_{ik}^{(1)} = A_i^-A_k^+ + B_i^-B_k^+$$
$$h^{(1)} = -\tfrac{1}{2}(\partial_1^2 + \partial_2^2) + \tfrac{1}{2}[(\partial_l w)^2 - \partial_l^2 w] + \tfrac{1}{2}\begin{pmatrix} \partial_1^2 w & \partial_1\partial_2 w \\ \partial_1\partial_2 w & \partial_2^2 w \end{pmatrix}$$
$$= -\frac{1}{2}(\partial_1^2 + \partial_2^2) + 72\alpha^2(y_1^2 + y_2^2) + 36\alpha\gamma + \frac{\gamma^2 - \sigma_3\gamma}{y_1^2} \tag{27}$$
$$+\frac{\gamma^2 - \frac{1}{2}\gamma\sigma_3 - \frac{\sqrt{3}}{2}\gamma\sigma_1}{(\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2)^2} + \frac{\gamma^2 - \frac{1}{2}\gamma\sigma_3 + \frac{\sqrt{3}}{2}\gamma\sigma_1}{(-\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2)^2}$$

where σ_i are Pauli matrices.

The above Hamiltonians $h^{(0)}$ and $h^{(2)}$ are intertwined with $h^{(1)}$:

$$\begin{aligned} h^{(0)}A_l^+ &= A_k^+h_{kl}^{(1)} \qquad & A_l^-h^{(0)} &= h_{lk}^{(1)}A_k^- \\ h^{(2)}B_l^+ &= B_k^+h_{kl}^{(1)} \qquad & B_l^-h^{(2)} &= h_{lk}^{(1)}B_k^- \qquad l, k = 1, 2. \end{aligned} \tag{28}$$

This chain of Hamiltonians $h^{(0)}$, $h_{lk}^{(1)}$ and $h^{(2)}$ determines the superHamiltonian as a Schrödinger-like operator with a 4×4 matrix potential of block-diagonal form. Intertwining relations (28) lead to interrelations between spectra and eigenfunctions of the chain Hamiltonians. Apart from possible zero modes of $A_l^\pm$, $B_l^\pm$, the spectrum of 2×2 matrix Hamiltonian $h^{(1)}$ is formed by two parts, coinciding with the spectra of the scalar Hamiltonians $h^{(0)}$ and $h^{(2)}$, correspondingly. Their eigenfunctions[10] are connected by the intertwining operators:

$$\begin{aligned} \Psi_k^{(1)}(E^{(0)}) &\sim A_k^-\Psi^{(0)}(E^{(0)}) \qquad & \Psi_k^{(1)}(E^{(2)}) &\sim B_k^-\Psi^{(2)}(E^{(2)}) \\ \Psi^{(0)}(E^{(0)}) &\sim A_k^+\Psi_k^{(1)}(E^{(0)}) \qquad & \Psi^{(2)}(E^{(2)}) &\sim B_k^+\Psi_k^{(1)}(E^{(2)}). \end{aligned} \tag{29}$$

Thus all (up to zero modes of the $A_l^\pm$, $B_l^\pm$) eigenvectors of the matrix Hamiltonian $h^{(1)}$ are expressed in terms of the Calogero wavefunctions.

In summary, we have used the framework of SUSY QM in order to derive an exactly solvable 2×2 matrix model, the spectrum of which is divided into two parts, each one coinciding with the spectrum of a scalar Calogero Hamiltonian. The reason why the spectrum of the matrix model (27) is still completely discrete can be found in the dominance of the confining

[10] $\Psi_l^{(1)}(E^{(0)})$ are the components $(l = 1, 2)$ of two-component vector eigenfunctions of the matrix Hamiltonian $h^{(1)}$.

scalar interaction over the coupling of internal degrees of freedom which is asymptotically decreasing. This matrix problem in a non-trivial way is related to a system of independent harmonic oscillators [25], but is not diagonalizable by standard transformations like rotations.

4. Time-dependent exactly solvable three-body matrix problems

In this section we will achieve the goal of obtaining scalar and matrix time-dependent exactly (quasi-exactly) solvable models and invariant operators. We start from general time-dependent intertwining relations which connect two time-dependent Schrödinger equations (TDSE)[11], one of them with a time-independent exactly solvable Hamiltonian. If $H(\vec{y})$ is an exactly solvable Hamiltonian and $H(\vec{y})\psi_n(\vec{y}) = E_n\psi_n(\vec{y})$, the intertwining relation with a known operator $U(\vec{y}, t)$:

$$(\mathrm{i}\partial_t - \tilde{H}(\vec{y}, t))U(\vec{y}, t) = U(\vec{y}, t)(\mathrm{i}\partial_t - H(\vec{y})) \tag{30}$$

leads to an exactly solvable time-dependent problem. All the solutions of

$$(\mathrm{i}\partial_t - \tilde{H}(\vec{y}, t))\tilde{\Psi}(\vec{y}, t) = 0$$

can be written as $U(\vec{y}, t)\Psi(\vec{y}, t)$, where $\Psi(\vec{y}, t) = \sum_{n=0}^{\infty} c_n \mathrm{e}^{-\mathrm{i}E_n t}\psi_n(\vec{y})$ is a generic time-dependent solution of the equation $(\mathrm{i}\partial_t - H(\vec{y}))\Psi(\vec{y}, t) = 0$.

For the one-dimensional case intertwining relations (30) were investigated in [19] for differential operators $U(y, t)$ of first and second order in derivatives. While in the one-dimensional problem a wide class of solutions was found, a straightforward extension to the two-dimensional case does not appear to be obvious. In this case it is more effective to study operators $U(\vec{y}, t)$ which can be written as products of two unitary pseudo-differential (of infinite order in derivatives) operators of the form [20]

$$U(\vec{y}, t) \equiv \exp\left\{\mathrm{i}a(t)\sum_i y_i^2\right\} \cdot \exp\left\{b(t)\sum_i (y_i\partial_i + \partial_i y_i)\right\} \tag{31}$$

where $a(t)$, $b(t)$ are arbitrary external time-dependent real functions. These operators have no zero modes. The intertwining relation (30) leads to

$$\tilde{H}(\vec{y}, t) = U(\vec{y}, t)H(\vec{y})U^{-1}(\vec{y}, t) + \mathrm{i}\left(\frac{\partial U(\vec{y}, t)}{\partial t}\right)U^{-1}(\vec{y}, t). \tag{32}$$

In the supersymmetric framework (sections 2 and 3) for each Hamiltonian of the chain one can choose the real valued coefficient functions $a^{(M)}(t)$, $b^{(M)}(t)$ independently for the different values of M. Under these unitary transformations $U^{(M)}$ the Jacobi canonical variables transform as

$$y_i \to U^{(M)}y_i(U^{(M)})^{-1} = y_i \cdot \exp\{2b^{(M)}(t)\}$$
$$p_i \equiv -\mathrm{i}\partial_i \to U^{(M)}p_i(U^{(M)})^{-1} = (p_i - 2a^{(M)}(t)y_i) \cdot \exp\{-2b^{(M)}(t)\}$$

and the so-called gauge term in (32) is

$$\mathrm{i}\left(\frac{\partial U^{(M)}(\vec{y}, t)}{\partial t}\right)(U^{(M)})^{-1}(\vec{y}, t) = (4a^{(M)}(t)\dot{b}^{(M)}(t) - \dot{a}^{(M)}(t)) \times \sum_i y_i^2 - \dot{b}^{(M)}(t)\sum_i (y_i p_i + p_i y_i). \tag{33}$$

After setting up the general framework of time-dependent intertwining of TDSE, we apply it to the Hamiltonians $h^{(M)}$ of the Calogero superchain of section 3. In particular, we identify

[11] Both Hamiltonians are assumed to be Hermitian.

$H(\vec{y})$ with the elements $h^{(M)}$ of the three-body Calogero chain $M = 0, 1, 2$ and generate a time-dependent chain. In general, the time-dependent Hamiltonians acquire new terms linear in momenta and time-dependent coefficients in all terms:

$$\begin{aligned}\tilde{h}^{(0)}(\vec{y},t) &= \tfrac{1}{2}\mathrm{e}^{-4b^{(0)}(t)}\sum_{i=1,2}p_i^2 - (a^{(0)}(t)\mathrm{e}^{-4b^{(0)}(t)} + \dot{b}^{(0)}(t))\sum_{i=1,2}(y_i p_i + p_i y_i)\\ &\quad +(2(a^{(0)}(t))^2\mathrm{e}^{-4b^{(0)}(t)} + 72\alpha^2\mathrm{e}^{4b^{(0)}(t)} + 4a^{(0)}(t)\dot{b}^{(0)}(t) - \dot{a}^{(0)}(t))\sum_{i=1,2}y_i^2\\ &\quad +\frac{1}{2}\mathrm{e}^{-4b^{(0)}(t)}\gamma(\gamma+1)\left[\frac{1}{y_1^2} + \frac{1}{(\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2)^2} + \frac{1}{(-\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2)^2}\right].\end{aligned} \tag{34}$$

The second scalar Hamiltonian $\tilde{h}^{(2)}(\vec{y},t)$ results from (26) with a similar construction.

The matrix Hamiltonian of the chain has the form

$$\begin{aligned}\tilde{h}^{(1)}(\vec{y},t) &= \tfrac{1}{2}\mathrm{e}^{-4b^{(1)}(t)}\sum_{i=1,2}p_i^2 - (a^{(1)}(t)\mathrm{e}^{-4b^{(1)}(t)} + \dot{b}^{(1)}(t))\sum_{i=1,2}(y_i p_i + p_i y_i)\\ &\quad +(2(a^{(1)}(t))^2\mathrm{e}^{-4b^{(1)}(t)} + 72\alpha^2\mathrm{e}^{4b^{(1)}(t)} + 4a^{(1)}(t)\dot{b}^{(1)}(t) - \dot{a}^{(1)}(t))\sum_{i=1,2}y_i^2 + 36\alpha\gamma\\ &\quad +\mathrm{e}^{-4b^{(1)}(t)}\left[\frac{\gamma^2 - \sigma_3\gamma}{y_1^2} + \frac{\gamma^2 - \frac{1}{2}\gamma\sigma_3 - \frac{\sqrt{3}}{2}\gamma\sigma_1}{(\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2)^2} + \frac{\gamma^2 - \frac{1}{2}\gamma\sigma_3 + \frac{\sqrt{3}}{2}\gamma\sigma_1}{(-\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2)^2}\right].\end{aligned} \tag{35}$$

The t dependence of the kinetic term can be interpreted as a t-dependent mass [21]. Linear terms in momenta are known to describe, for example, the coupling of charged particles with gauge potentials and therefore have not to be discarded *a priori*. However, the terms linearly dependent on momenta drop out for a particular relation:

$$a(t) = -\dot{b}(t)\exp(4b(t)). \tag{36}$$

We remark that, in the case of the factorization of $\tilde{h}^{(M)}(\vec{y},t) = \eta(t)h^{(M)}(\vec{y})$, TDSE reduces effectively to a quasi-stationary problem, because by a suitable reparametrization of time $t \to \tau \equiv \int \eta(t)\,\mathrm{d}t$ the problem becomes stationary. The corresponding constraint leads for $M = 0, 1, 2$ again to (36) and to a nonlinear differential equation for $b(t)$:

$$\ddot{b}(t) + 6\dot{b}^2(t) + 72\alpha^2(\mathrm{e}^{-8b(t)} - 1) = 0. \tag{37}$$

The general solution of this equation involves elliptic integrals in the relation between t and b. The function $\eta(t)$ becomes $\eta(t) = \exp(-4b(t))$.

The construction of invariant operators R, which satisfy the equation

$$\frac{\partial R}{\partial t} + \mathrm{i}[\tilde{H}(\vec{y},t), R] = 0 \tag{38}$$

is an important aspect of the investigation of time-dependent systems [26]. In our framework from the intertwining relation (30) the invariant operator exists and can be expressed in terms of $h^{(M)}$ and $U^{(M)}$:

$$\begin{aligned}R^{(M)}(t) &\equiv U^{(M)}(\vec{y},t)h^{(M)}(\vec{y})(U^{(M)})^{-1}(\vec{y},t)\\ &= \tilde{h}^{(M)}(\vec{y},t) - \mathrm{i}\left(\frac{\partial U^{(M)}(\vec{y},t)}{\partial t}\right)(U^{(M)})^{-1}(\vec{y},t)\end{aligned} \tag{39}$$

where the last term is usually referred as a gauge term.

The invariant operator is Hermitian because the intertwining operator $U(\vec{y},t)$ is unitary. From the equations (33)–(35) it is straightforward to obtain the explicit expression for the chain of invariants of this model. In particular, one can notice that $R^{(M)}$ still have the structure similar to the Calogero Hamiltonians (34) and (35), though some terms are missing.

In general, one can argue from the similarity (39) that the spectrum of $R^{(M)}$ is the same as the spectrum of $h^{(M)}$ and therefore time-independent [27]. The operators $R^{(M)}$ provide an additional exactly solvable (matrix and scalar) models with explicit time-dependent potentials but with time-independent spectra. Their eigenfunctions depend parametrically on time via $U^{(M)}(\vec{y}, t)$ applied to the stationary eigenfunctions of $h^{(M)}$. Let us remark that invariant operators $R(t)$ admit a quasi-factorization like (24) in section 3 with suitable (transformed by $U(\vec{y}, t)$) components of supercharge, but $\tilde{h}(\vec{y}, t)$ do not because of the gauge term.

5. Conclusions

Given for granted the usefulness of exactly (and quasi-exactly) solvable models we would like to point out that our contribution has been to construct explicitly few models of such a kind with a discrete spectrum: among them the exactly solvable three-particle (matrix and scalar) non-stationary Calogero models and quasi-exactly-solvable N-particle matrix stationary models. An extension of the method of section 4 to the N-body Calogero model described in section 3 leads to time-dependent quasi-exactly-solvable matrix models. Since it is not usual to find exactly solvable or quasi-exactly-solvable time-dependent problems, especially in a context of many-body systems, our results support the program to investigate further time-dependent generalizations of stationary solvable models, such as those mentioned in section 1 [2–4, 6] and quasi-exactly-solvable matrix models [28]. In particular, one can focus attention on the dynamical algebras of these models [29] to construct Ermakov–Lewis invariant operators ([26] and references therein). A less straightforward task will be to modify the model in such a way as to allow for coexistence [30] of a continuum and a discrete spectrum describing scattering and bound states.

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